

Are all classical superintegrable systems in two-dimensional space linearizable?

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Abstract

Several examples of classical superintegrable systems in two-dimensional space are shown to possess hidden symmetries leading to their linearization. They are those determined 50 years ago in [1], and the more recent Tremblay-Turbiner-Winternitz system [6]. We conjecture that all classical superintegrable systems in two-dimensional space have hidden symmetries that make them linearizable.

1 Introduction

About 50 years ago in a seminal paper [1], the authors considered Hamiltonian systems with Hamiltonian given either in cartesian coordinates, i.e.:

$$H = \frac{1}{2} (p_1^2 + p_2^2) + V(x_1, x_2), \quad (1)$$

or in polar coordinates, i.e.:

$$H = \frac{1}{2} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) + V(r, \varphi), \quad (2)$$

Their purpose was to determine all the potentials such that the corresponding Hamiltonian system admits two first integrals that are quadratic in the momenta, in addition to the Hamiltonian. No assumption about the separation of variables in the Hamilton-Jacobi equation was made a priori. Four independent potentials were found and it was proven

that the corresponding Hamilton-Jacobi equation was separable in at least two different coordinate systems. Two of the four potentials were given in cartesian coordinates:

$$V_I(x_1, x_2) = \frac{\omega^2}{2}(x_1^2 + x_2^2) + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}, \quad (3a)$$

$$V_{II}(x_1, x_2) = \frac{\omega^2}{2}(4x_1^2 + x_2^2) + \beta_1 x_1 + \frac{\beta_2}{x_2^2}, \quad (3b)$$

while the other two were given in polar coordinates:

$$V_{III}(r, \varphi) = \frac{\alpha}{r} + \frac{1}{r^2} \left(\frac{\beta_1}{\cos^2\left(\frac{\varphi}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{\varphi}{2}\right)} \right) \quad (3c)$$

$$V_{IV}(r, \varphi) = \frac{\alpha}{r} + \frac{1}{\sqrt{r}} \left(\beta_1 \cos\left(\frac{\varphi}{2}\right) + \beta_2 \sin\left(\frac{\varphi}{2}\right) \right). \quad (3d)$$

These four cases belong to the class of two-dimensional superintegrable systems, namely those Hamiltonian systems that admit three first integrals. Actually they are also maximally superintegrable. In fact a Hamiltonian system with n degrees of freedom is called superintegrable if allows $n + 1$ integrals, and maximally superintegrable if the integrals are $2n - 1$. For $n = 2$ the two definitions coincide.

In any undergraduate text of Mechanics, e.g. [2], it is shown that Kepler problem in polar coordinates is linearizable, namely that one can exactly transform its nonlinear equations of motion into the equation of an harmonic linear oscillator. In [3] it was shown that such a linearization can be achieved by means of the reduction method that was proposed in [4] in order to find hidden symmetries of Kepler problem. Moreover, the reduction method was successfully applied to generalizations of the Kepler problem with and without drag in order to find their hidden linearity, although not all of them admit a Lagrangian description [5].

In 2009 a new two-dimensional superintegrable system was determined [6], and it has been known since as the Tremblay-Turbiner-Winternitz (TTW) system.

In 2011 a two-dimensional superintegrable system such that the corresponding Hamilton-Jacobi equation does not admit separation of variables in any coordinates was studied in [7]. In [8] it was found that its Lagrangian equations can be transformed into a linear third-order equation by applying the reduction method [4].

In the present paper we show that the Lagrangian equations corresponding to the potentials V_I, V_{II}, V_{III} , and some of their generalizations are all linearizable by means of their hidden symmetries. We also prove that the TTW system is linearizable.

The Hamiltonian system with potential V_{IV} is a subcase of the linearizable cases determined in [3], where the following Newtonian equations were considered:

$$\ddot{r} - r\dot{\varphi}^2 + g = 0, \quad (4)$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + h = 0, \quad (5)$$

with

$$g = \frac{U''(\varphi) + U(\varphi)}{r^2} + 2\frac{W'(\varphi)}{r^{3/2}}, \quad h = \frac{W(\varphi)}{r^{3/2}}. \quad (6)$$

It corresponds to the substitution [5]:

$$U = \alpha, \quad W = \frac{1}{2} \left(\beta_1 \sin\left(\frac{\varphi}{2}\right) - \beta_2 \cos\left(\frac{\varphi}{2}\right) \right). \quad (7)$$

See also [9].

All the superintegrable systems that we consider here are in real Euclidean space. In a forthcoming paper [10], we will show that many known superintegrable systems in space of non-constant curvature are also linearizable, e.g. the three superintegrable systems for the Darboux space of Type I determined in [11].

We conclude with a conjecture, namely that all two-dimensional superintegrable systems are linearizable by means of their hidden symmetries.

2 Linearity of the Lagrangian equations with potentials V_I , V_{II} , and V_{III}

The Lagrangian corresponding to the Hamiltonian (1) in cartesian coordinates is:

$$L = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2), \quad (8)$$

while the Lagrangian corresponding to the Hamiltonian (2) in polar coordinates is

$$L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r, \varphi). \quad (9)$$

Remark 1: We have applied Douglas' method [12] to the Lagrangian equations corresponding to the four potentials. The two potentials V_I and V_{II} lead to many different Lagrangians, while in the case of potentials V_{III} and V_{IV} , there exists only one Lagrangian in analogy with Kepler's problem. ■

2.1 The potential V_I

The Lagrangian equations corresponding to the Lagrangian (8) with $V = V_I$ are:

$$\ddot{x}_1 = -\omega^2 x_1 + \frac{2\beta_1}{x_1^3},$$

$$\ddot{x}_2 = -\omega^2 x_2 + \frac{2\beta_2}{x_2^3}. \quad (10)$$

This Lagrangian admits three Noether symmetries generated by the operators:

$$\begin{aligned} \Sigma_1 &= \partial_t, \quad \Sigma_2 = \cos(2\omega t)\partial_t - \omega x_1 \sin(2\omega t)\partial_{x_1} - \omega x_2 \sin(2\omega t)\partial_{x_2}, \\ \Sigma_3 &= \sin(2\omega t)\partial_t + \omega x_1 \cos(2\omega t)\partial_{x_1} + \omega x_2 \cos(2\omega t)\partial_{x_2}. \end{aligned} \quad (11)$$

which correspond to the algebra $sl(2, \mathbb{R})$. The application of Noether's theorem yields three first integrals. From Σ_1 comes the Hamiltonian, i.e.:

$$H_I = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{\omega^2}{2}(x_1^2 + x_2^2) + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}. \quad (12)$$

and from Σ_2 and Σ_3 the following two time-dependent integrals:

$$\begin{aligned} K_2 &= \left[\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} + \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{\omega^2}{2}(x_1^2 + x_2^2) \right] \cos(2\omega t) \\ &\quad + \omega(x_1 \dot{x}_1 + x_2 \dot{x}_2) \sin(2\omega t), \end{aligned} \quad (13)$$

and

$$\begin{aligned} K_3 &= \left[\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} + \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{\omega^2}{2}(x_1^2 + x_2^2) \right] \sin(2\omega t) \\ &\quad - \omega(x_1 \dot{x}_1 + x_2 \dot{x}_2) \cos(2\omega t), \end{aligned} \quad (14)$$

respectively.

Remark 2: Another time-independent first integral can be obtained by the following ubiquitous combination:

$$H_I^2 - K_2^2 - K_3^2 = \omega^2 \left(2\beta_1 + 2\beta_2 + 2\beta_1 \frac{x_2^2}{x_1^2} + 2\beta_2 \frac{x_1^2}{x_2^2} + (x_2 \dot{x}_1 - x_1 \dot{x}_2)^2 \right) \quad (15)$$

Such a combination can be found in other instances where a couple of time-dependent first integrals are derived from Noether's theorem. ■

The presence of the algebra $sl(2, \mathbb{R})$ suggests to eliminate the two parameters β_1 and β_2 by raising the order, as it was done in [13] in the case of the isotonic oscillator. We solve system (10) with respect to β_1 and β_2 , i.e.:

$$\beta_1 = \frac{1}{2}(x_1^3 \ddot{x}_1 + \omega^2 x_1^4),$$

$$\beta_2 = \frac{1}{2}(x_2^3 \ddot{x}_2 + \omega^2 x_2^4), \quad (16)$$

and then we differentiate them with respect to t in order to get the following two third-order equations:

$$\begin{aligned} \ddot{x}_1 &= -\frac{\dot{x}_1}{x_1}(4\omega^2 x_1 + 3\ddot{x}_1), \\ \ddot{x}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega^2 x_2 + 3\ddot{x}_2). \end{aligned} \quad (17)$$

This system admits a thirteen-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{aligned} \Gamma_1 &= \cos(2\omega t)\partial_t - \sin(2\omega t)\omega x_1\partial_{x_1} - \sin(2\omega t)\omega x_2\partial_{x_2}, \\ \Gamma_2 &= \sin(2\omega t)\partial_t + \cos(2\omega t)\omega x_1\partial_{x_1} + \cos(2\omega t)\omega x_2\partial_{x_2}, \\ \Gamma_3 &= \partial_t, \\ \Gamma_4 &= \frac{\cos(2\omega t)}{x_1}\partial_{x_1}, \\ \Gamma_5 &= \frac{\sin(2\omega t)}{x_1}\partial_{x_1}, \\ \Gamma_6 &= \frac{\cos(2\omega t)}{x_2}\partial_{x_2}, \\ \Gamma_7 &= \frac{\sin(2\omega t)}{x_2}\partial_{x_2}, \\ \Gamma_8 &= \frac{x_2^2}{x_1}\partial_{x_1}, \\ \Gamma_9 &= x_1\partial_{x_1}, \\ \Gamma_{10} &= \frac{1}{x_1}\partial_{x_1}, \\ \Gamma_{11} &= \frac{x_1^2}{x_2}\partial_{x_2}, \\ \Gamma_{12} &= x_2\partial_{x_2}, \\ \Gamma_{13} &= \frac{1}{x_2}\partial_{x_2}. \end{aligned} \quad (18)$$

Therefore system (17) is linearizable. In order to find the linearizing transformation we could use the method in [14,15] based on the classification of the four-dimensional Abelian subalgebras [16]. Instead we recall that the following linear system¹:

$$\ddot{u}_1 = -4\omega^2 \dot{u}_1,$$

¹It is the derivative with respect to t of the equations of a two-dimensional isotropic oscillator with frequency 2ω .

$$\ddot{u}_2 = -4\omega^2 \dot{u}_2, \quad (19)$$

admits a thirteen-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{aligned} \bar{\Gamma}_1 &= \cos(2\omega t) \partial_t - 2 \sin(2\omega t) \omega u_1 \partial_{u_1} - 2 \sin(2\omega t) \omega u_2 \partial_{u_2}, \\ \bar{\Gamma}_2 &= \sin(2\omega t) \partial_t + 2 \cos(2\omega t) \omega u_1 \partial_{u_1} + 2 \cos(2\omega t) \omega u_2 \partial_{u_2}, \\ \bar{\Gamma}_3 &= \partial_t, \\ \bar{\Gamma}_4 &= \cos(2\omega t) \partial_{u_1}, \\ \bar{\Gamma}_5 &= \sin(2\omega t) \partial_{u_1}, \\ \bar{\Gamma}_6 &= \cos(2\omega t) \partial_{u_2}, \\ \bar{\Gamma}_7 &= \sin(2\omega t) \partial_{u_2}, \\ \bar{\Gamma}_8 &= u_2 \partial_{u_1}, \\ \bar{\Gamma}_9 &= u_1 \partial_{u_1}, \\ \bar{\Gamma}_{10} &= \partial_{u_1}, \\ \bar{\Gamma}_{11} &= u_1 \partial_{u_2}, \\ \bar{\Gamma}_{12} &= u_2 \partial_{u_2}, \\ \bar{\Gamma}_{13} &= \partial_{u_2}. \end{aligned} \quad (20)$$

Consequently, if we make the following transformation:

$$u_1 = \frac{x_1^2}{2}, \quad u_2 = \frac{x_2^2}{2} \quad (21)$$

system (10) becomes the linear system (19).

More recently the following generalization of the potential V_I has been proposed and proved superintegrable [18–21]:

$$V_I^{\text{gen}}(x_1, x_2) = \frac{\omega_1^2}{2} x_1^2 + \frac{\omega_2^2}{2} x_2^2 + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}. \quad (22)$$

Applying the same procedure as described above to the corresponding Lagrangian equations, i.e.:

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2 x_1 + \frac{2\beta_1}{x_1^3}, \\ \ddot{x}_2 &= -\omega_2^2 x_2 + \frac{2\beta_2}{x_2^3}, \end{aligned} \quad (23)$$

yields the following system of two third-order equations:

$$\begin{aligned}\ddot{x}_1 &= -\frac{\dot{x}_1}{x_1}(4\omega_1^2 x_1 + 3\ddot{x}_1), \\ \ddot{x}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega_2^2 x_2 + 3\ddot{x}_2),\end{aligned}\tag{24}$$

which admits a nine-dimensional Lie point symmetry algebra generated by the operators $\Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8, \Gamma_{10}, \Gamma_{11}, \Gamma_{13}$ in (18). Indeed by applying again the transformation (21) we obtain that the system (24) is transformed into the following linear system:

$$\begin{aligned}\ddot{u}_1 &= -4\omega_1^2 \dot{u}_1, \\ \ddot{u}_2 &= -4\omega_2^2 \dot{u}_2,\end{aligned}\tag{25}$$

namely the derivative with respect to t of the equations of a two-dimensional anisotropic oscillator.

2.2 The potential V_{II}

The Lagrangian equations corresponding to the Lagrangian (8) with $V = V_{\text{II}}$ are:

$$\ddot{x}_1 = -4\omega^2 x_1 - \beta_1, \tag{26a}$$

$$\ddot{x}_2 = -\omega^2 x_2 + \frac{2\beta_2}{x_2^3}. \tag{26b}$$

This Lagrangian admits three Noether symmetries generated by the following operators:

$$\Upsilon_1 = \partial_t, \quad \Upsilon_2 = \sin(2\omega t)\partial_{x_1}, \quad \Upsilon_3 = \cos(2\omega t)\partial_{x_2}, \tag{27}$$

that is the algebra $A_{3,6} \simeq \langle \Upsilon_1/(2\omega), \Upsilon_3, \Upsilon_2 \rangle$ in the classification given in [16]. The application of Noether's theorem yields three first integrals. From Υ_1 comes the Hamiltonian, i.e.:

$$H_{\text{II}} = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{\omega^2}{2}(4x_1^2 + x_2^2) + \beta_1 x_1 + \frac{\beta_2}{x_2^2} \tag{28}$$

and from Υ_2 and Υ_3 the following two time-dependent integrals:

$$Y_2 = \cos(2\omega t)\beta_1 + 4\cos(2\omega t)\omega^2 x_1 - 2\sin(2\omega t)\omega \dot{x}_1, \tag{29}$$

and

$$Y_3 = -2\cos(2\omega t)\omega \dot{x}_1 - \sin(2\omega t)\beta_1 - 4\sin(2\omega t)\omega^2 x_1, \tag{30}$$

respectively.

Remark 3: The following combination of (29) and (30) yields the Hamiltonian for the equation (26a) only, i.e.:

$$H_1 = \frac{Y_2^2 + Y_3^2}{8\omega^2} = \frac{1}{2}\dot{x}_1^2 + 2\omega^2 x_1^2 + \beta_1 x_1 + \frac{\beta_1^2}{8\omega^2}. \quad (31)$$

The following combination of (31) and the Hamiltonian (28) yields the Hamiltonian for the equation (26b) only, i.e.:

$$H_2 = H_{\text{II}} - \frac{Y_2^2 + Y_3^2}{8\omega^2} = \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}\omega^2 x_2^2 + \frac{\beta_2}{x_2^2} - \frac{\beta_1^2}{8\omega^2}. \quad (32)$$

Of course, the addition/sottraction of the constant $\frac{\beta_1^2}{8\omega^2}$ does not influence either the Hamiltonian H_1 or H_2 . ■

We solve system (26a)-(26b) with respect to β_1 and β_2 , i.e.:

$$\begin{aligned} \beta_1 &= -\ddot{x}_1 - 4\omega^2 x_1, \\ \beta_2 &= \frac{1}{2}(x_2^3 \ddot{x}_2 + \omega^2 x_2^4), \end{aligned} \quad (33)$$

and then we take the derivative with respect to t in order to get the following system of two third-order equations:

$$\begin{aligned} \ddot{\ddot{x}}_1 &= -4\omega^2 \dot{x}_1, \\ \ddot{\ddot{x}}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega^2 x_2 + 3\ddot{x}_2). \end{aligned} \quad (34)$$

It should not be a surprise that this system admits a thirteen-dimensional Lie symmetry algebra. Consequently, the transformation

$$u_1 = x_1, \quad u_2 = \frac{x_2^2}{2} \quad (35)$$

takes system (34) into the linear system (19), namely that obtained by taking the derivative with respect to t of the equations of a two-dimensional isotropic oscillator with frequency 2ω .

2.3 The potential V_{III}

The Lagrangian equations corresponding to the Lagrangian (9) with $V = V_{\text{III}}$ are:

$$\begin{aligned}\ddot{r} &= \frac{\alpha}{r^2} + r\dot{\varphi}^2 + \frac{2}{r^3} \left(\frac{\beta_1}{\cos^2(\frac{\varphi}{2})} + \frac{\beta_2}{\sin^2(\frac{\varphi}{2})} \right), \\ \ddot{\varphi} &= -\frac{2}{r}\dot{r}\dot{\varphi} - \frac{1}{r^4} \left(\frac{\beta_1 \sin(\frac{\varphi}{2})}{\cos^3(\frac{\varphi}{2})} - \frac{\beta_2 \cos(\frac{\varphi}{2})}{\sin^3(\frac{\varphi}{2})} \right).\end{aligned}\quad (36)$$

This Lagrangian admits one Noether symmetry, i.e. translation in t , and Noether theorem yields the Hamiltonian.

We now write the two second-order Lagrangian equations (36) as the following four first-order equations

$$\begin{aligned}\dot{w}_1 &= w_3, \\ \dot{w}_2 &= w_4, \\ \dot{w}_3 &= \frac{\alpha}{w_1^2} + w_1 w_4^2 + \frac{2}{w_1^3} \left(\frac{\beta_1}{\cos^2(\frac{w_2}{2})} + \frac{\beta_2}{\sin^2(\frac{w_2}{2})} \right), \\ \dot{w}_4 &= -\frac{2}{w_1} w_3 w_4 - \frac{1}{w_1^4} \left(\frac{\beta_1 \sin(\frac{w_2}{2})}{\cos^3(\frac{w_2}{2})} - \frac{\beta_2 \cos(\frac{w_2}{2})}{\sin^3(\frac{w_2}{2})} \right),\end{aligned}\quad (37)$$

with the identification

$$(w_1, w_2, w_3, w_4) \equiv (r, \varphi, \dot{r}, \dot{\varphi}). \quad (38)$$

We apply the reduction method [4] by choosing $w_2 = y$ as the new independent variable, and consequently the following system of three first-order equations is obtained:

$$\frac{dw_1}{dy} = \frac{w_3}{w_4}, \quad (39)$$

$$\frac{dw_3}{dy} = \frac{\alpha}{w_1^2 w_4} + w_1 w_4 + \frac{2}{w_1^3 w_4} \left(\frac{\beta_1}{\cos^2(\frac{y}{2})} + \frac{\beta_2}{\sin^2(\frac{y}{2})} \right), \quad (40)$$

$$\frac{dw_4}{dy} = -\frac{2}{w_1} w_3 - \frac{1}{w_1^4 w_4} \left(\frac{\beta_1 \sin(\frac{y}{2})}{\cos^3(\frac{y}{2})} - \frac{\beta_2 \cos(\frac{y}{2})}{\sin^3(\frac{y}{2})} \right), \quad (41)$$

We derive w_3 from equation (39), i.e.,

$$w_3 = w_4 \frac{dw_1}{dy}, \quad (42)$$

and consequently equation (41) becomes:

$$\frac{dw_4}{dy} + \frac{2w_4}{w_1} \frac{dw_1}{dy} + \frac{1}{w_1^4 w_4} \left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos^3\left(\frac{y}{2}\right)} - \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin^3\left(\frac{y}{2}\right)} \right) = 0, \quad (43)$$

which can be simplified by means of the following transformation, i.e.:

$$w_4 = \frac{r_4}{w_1^2}, \quad (44)$$

with r_4 a new function of y that then has to satisfy the following equation:

$$\frac{dr_4}{dy} = -\frac{1}{r_4} \left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos^3\left(\frac{y}{2}\right)} - \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin^3\left(\frac{y}{2}\right)} \right). \quad (45)$$

Its general solution is easily obtained to be:

$$r_4 = \pm \sqrt{a_1 - 2 \left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos\left(\frac{y}{2}\right)} + \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)} \right)}, \quad (46)$$

with a_1 an arbitrary constant. Finally, equation (40) becomes the following second-order differential equation:

$$\begin{aligned} \frac{d^2 w_1}{dy^2} = & \frac{2}{w_1} \left(\frac{dw_1}{dy} \right)^2 + \frac{w_1(\alpha w_1 + a_1) \sin^2\left(\frac{y}{2}\right) \cos^2\left(\frac{y}{2}\right)}{a_1 \sin^2\left(\frac{y}{2}\right) \cos^2\left(\frac{y}{2}\right) - 2\beta_1 \sin^2\left(\frac{y}{2}\right) - 2\beta_2 \cos^2\left(\frac{y}{2}\right)} \\ & + \frac{\beta_1 \sin^4\left(\frac{y}{2}\right) - \beta_2 \cos^4\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right) \cos\left(\frac{y}{2}\right) (a_1 \sin^2\left(\frac{y}{2}\right) \cos^2\left(\frac{y}{2}\right) - 2\beta_1 \sin^2\left(\frac{y}{2}\right) - 2\beta_2 \cos^2\left(\frac{y}{2}\right))} \frac{dw_1}{dy}. \end{aligned} \quad (47)$$

This equation admits an eight-dimensional Lie point symmetry algebra, which means that it is linearizable. The linearizing transformation is obtained by means of Lie's canonical representation of a two-dimensional abelian intransitive subalgebra [17]. One such subalgebra is that generated by the following two operators:

$$\begin{aligned} \Xi_1 &= (2\beta_1 - 2\beta_2 - \cos(y)a_1)w_1^2 \partial_{w_1}, \\ \Xi_2 &= \sqrt{4\beta_2 \cos(y) - 4\beta_1 \cos(y) + a_1 \cos^2(y) + 4\beta_1 + 4\beta_2 - a_1 w_1^2} \partial_{w_1}, \end{aligned} \quad (48)$$

that we have to put in the canonical form $\partial_{\tilde{w}_1}, \tilde{y} \partial_{\tilde{w}_1}$. Therefore the transformation

$$\tilde{y} = -\frac{\sqrt{4\beta_2 \cos(y) - 4\beta_1 \cos(y) + a_1 \cos^2(y) + 4\beta_1 + 4\beta_2 - a_1}}{-2\beta_1 + 2\beta_2 + \cos(y)a_1}, \quad (49)$$

$$\tilde{w}_1 = \frac{1}{(-2\beta_1 + 2\beta_2 + \cos(y)a_1)w_1} \quad (50)$$

takes equation (47) into a linear equation of the type

$$\frac{d^2 \tilde{w}_1}{d\tilde{y}^2} = \mathfrak{F}(\tilde{y}). \quad (51)$$

Actually (50) suggests the simpler transformation

$$u = \frac{1}{w_1}, \quad (52)$$

that applied to equation (47) yields the following linear equation:

$$\ddot{u} = \frac{\dot{u} \left(\beta_1 \sin^4 \left(\frac{y}{2} \right) - \beta_2 \cos^4 \left(\frac{y}{2} \right) \right) - (\alpha - a_1 u) \sin^3 \left(\frac{y}{2} \right) \cos^3 \left(\frac{y}{2} \right)}{\left(a_1 \sin^2 \left(\frac{y}{2} \right) \cos^2 \left(\frac{y}{2} \right) - 2\beta_1 \sin^2 \left(\frac{y}{2} \right) - 2\beta_2 \cos^2 \left(\frac{y}{2} \right) \right) \sin \left(\frac{y}{2} \right) \cos \left(\frac{y}{2} \right)}. \quad (53)$$

There exists a generalization of the potential V_{III} , i.e.:

$$V_{\text{III}}^{\text{gen}}(r, \varphi) = \frac{\alpha}{r} + \frac{f(\varphi)}{r^2}, \quad (54)$$

where f is an arbitrary function of φ . Then the corresponding equations are:

$$\ddot{r} = r\dot{\varphi}^2 + \frac{\alpha}{r^2} + \frac{2f(\varphi)}{r^3}, \quad (55)$$

$$\ddot{\varphi} = -2\frac{\dot{r}}{r}\dot{\varphi} - \frac{f'(\varphi)}{r^2}, \quad (56)$$

where prime indicates the derivative of f with respect to φ . Introducing the new variables w_1, w_2, w_3, w_4 as in (38) yields the following Hamilton equations:

$$\dot{w}_1 = w_3, \quad (57)$$

$$\dot{w}_2 = \frac{w_4}{w_1^2}, \quad (58)$$

$$\dot{w}_3 = \frac{\alpha w_1 + 2f(w_2) + w_4^2}{w_1^3}, \quad (59)$$

$$\dot{w}_4 = -\frac{f'(w_2)}{w_1^2}. \quad (60)$$

We apply the reduction method [4] by choosing $w_2 = y$ as the new independent variable, and consequently the following system of three first-order equations is obtained:

$$\frac{dw_1}{dy} = \frac{w_3 w_1^2}{w_4}, \quad (61)$$

$$\frac{dw_3}{dy} = \frac{\alpha w_1 + 2f(y) + w_4^2}{w_1 w_4}, \quad (62)$$

$$\frac{dw_4}{dy} = -\frac{f'(y)}{w_4}. \quad (63)$$

Equation (63) can be integrated to give

$$w_4 = \pm \sqrt{J - 2f(y)}, \quad (64)$$

with J an arbitrary constant. Finally, eliminating w_3 by means of (61), i.e.

$$w_3 = \frac{w_4}{w_1^2} \frac{dw_1}{dy}, \quad (65)$$

yields the following second-order equation for $w_1 = w_1(y)$:

$$\frac{d^2 w_1}{dy^2} = \frac{2(J - 2f(y)) \left(\frac{dw_1}{dy} \right)^2 + f'(y) w_1 \frac{dw_1}{dy} + \alpha w_1^3 + w_1^2 J}{w_1 (J - 2f(y))}. \quad (66)$$

This equation is linearizable since it admits an eight-dimensional Lie symmetry algebra. As in the case of equation (47), the transformation (52) yields the linear equation:

$$\ddot{u} = \frac{f'(y)\dot{u} - Ju - \alpha}{J - 2f(y)}. \quad (67)$$

3 The Tremblay-Turbiner-Winternitz system

We now consider the superintegrable Tremblay-Turbiner-Winternitz (TTW) system [6], namely an Hamiltonian system with a potential that generalizes V_I in (3a), i.e.:

$$V_{\text{TTW}}(r, \varphi) = \omega^2 r^2 + \frac{k^2}{r^2} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right). \quad (68)$$

The corresponding Lagrangian, i.e.:

$$L_{\text{TTW}} = \frac{1}{4} (\dot{r}^2 + r^2 \dot{\varphi}^2) - \omega^2 r^2 - \frac{k^2}{r^2} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right), \quad (69)$$

yields the following Lagrangian equations:

$$\ddot{r} = -4\omega^2 r + r\dot{\varphi}^2 + \frac{4k^2}{r^3} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right), \quad (70a)$$

$$\ddot{\varphi} = -\frac{2}{r}\dot{r}\dot{\varphi} - \frac{4k^3}{r^4} \left(\frac{\beta_1 \sin(k\varphi)}{\cos^3(k\varphi)} - \frac{\beta_2 \cos(k\varphi)}{\sin^3(k\varphi)} \right), \quad (70b)$$

that admit a three-dimensional Lie point symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$ spanned by:

$$\begin{aligned} \Sigma_1 &= \partial_t, & \Sigma_2 &= \cos(4\omega t)\partial_t - 2\omega \sin(4\omega t)r\partial_r, \\ \Sigma_3 &= \sin(4\omega t)\partial_t + 2\omega \cos(4\omega t)r\partial_r, \end{aligned} \quad (71)$$

which are also Noether symmetries of the Lagrangian (69). The application of Noether's theorem yields three first integrals, one being the Hamiltonian, i.e.:

$$H_{TTW} = \frac{1}{4}(\dot{r}^2 + r^2\dot{\varphi}^2) + \omega^2 r^2 + \frac{k^2}{r^2} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right). \quad (72)$$

The other two first integrals depend on t , i.e.:

$$\begin{aligned} K_{2TTW} &= \left[\frac{1}{4}(\dot{r}^2 + r^2\dot{\varphi}^2) - \omega^2 r^2 + \frac{k^2}{r^2} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right) \right] \cos(4\omega t) \\ &\quad + \omega r \dot{r} \sin(4\omega t), \end{aligned} \quad (73)$$

$$\begin{aligned} K_{3TTW} &= \left[\frac{1}{4}(\dot{r}^2 + r^2\dot{\varphi}^2) - \omega^2 r^2 + \frac{k^2}{r^2} \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right) \right] \sin(4\omega t) \\ &\quad - \omega r \dot{r} \cos(4\omega t). \end{aligned} \quad (74)$$

Remark 4: Another time-independent first integral can be obtained by the following combination:

$$H_{TTW}^2 - K_{2TTW}^2 - K_{3TTW}^2 = r^4\dot{\varphi}^2 + 4k^2 \left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)} \right). \quad (75)$$

■

The presence of the algebra $\mathfrak{sl}(2, \mathbb{R})$ suggests to eliminate the two parameters β_1 and β_2 by raising the order. We solve system (70) with respect to β_1 and β_2 , and then we take the derivative with respect to t which yields the following two third-order equations:

$$r\ddot{r}\ddot{r} + \dot{r}(16\omega^2 r + 3\ddot{r}) = 0, \quad (76)$$

$$\begin{aligned} &\cos(k\varphi)\sin(k\varphi)r^2\ddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} + 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 \\ &+ 16\cos(k\varphi)\sin(k\varphi)k^2\omega^2 r^2\dot{\varphi} - 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2r\ddot{r}\dot{\varphi} \\ &\quad + 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi} \\ &+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \end{aligned} \quad (77)$$

Equation (76) admits a seven-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned} X_1 &= \partial_t, \quad X_2 = \cos(4\omega t)\partial_t - 2\omega \sin(4\omega t)r\partial_r, \quad X_3 = \sin(4\omega t)\partial_t + 2\omega \cos(4\omega t)r\partial_r, \\ X_4 &= \frac{\cos(4\omega t)}{r}\partial_r, \quad X_5 = \frac{\sin(4\omega t)}{r}\partial_r, \quad X_6 = r\partial_r, \quad X_7 = \frac{1}{r}\partial_r, \end{aligned} \quad (78)$$

and consequently it is linearizable. We find that a two-dimensional non-abelian intransitive subalgebra is that generated by X_6 and X_7 , and following Lie's classification [17], if we transform them into their canonical form, i.e., $\partial_u, u\partial_u$, then we obtain that the new dependent variable is given by

$$u = \frac{r^2}{2}$$

and consequently equation (76) becomes

$$\ddot{u} = -16\omega^2 \dot{u}, \quad (79)$$

namely the derivative with respect to t of the equation of a linear harmonic oscillator with frequency 4ω . Thus, the general solution of (76) is

$$r = \sqrt{a_1 + a_2 \cos(4\omega t) + a_3 \sin(4\omega t)}. \quad (80)$$

Equation (77) is also linearizable since it admits a seven-dimensional Lie symmetry algebra generated by

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi, \quad (81)$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\ddot{\ddot{s}}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}rk^2s_1 \quad (82a)$$

$$+ 8\ddot{r}\dot{s}_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0,$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{s}_1 - 2\dot{r}^2s_1 = 0, \quad (82b)$$

$$r^2\ddot{\ddot{s}}_3 + 6\dot{s}_3\ddot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0, \quad (82c)$$

with r given in (80). Similarly to equation (76), we find that a two-dimensional non-abelian intransitive subalgebra is generated by the operators

$$-\frac{1}{2k}\cot(k\varphi)\partial_\varphi, \quad \frac{2}{\sin(2k\varphi)}\partial_\varphi, \quad (83)$$

that put into canonical form yield the new dependent variable

$$v = -\frac{1}{2k}\cos^2(k\varphi),$$

and consequently equation (77) becomes linear, i.e.:

$$\ddot{v} = -\frac{6\dot{r}}{r}\dot{v} - \frac{2}{r^2} (3\dot{r}^2 + 8k^2\omega^2 r^2 + (2k^2 + 1)r\ddot{r}) \dot{v}. \quad (84)$$

Remark 5: The TTW system admits closed orbits if k is rational, as it has been shown by various methods in [22], [23], [24]. We observe that equation (84) yields solutions of (70) in terms of hypergeometric and trigonometric functions if k is rational, although the linearization that we have achieved remains valid even for k irrational. ■

4 Conclusions

In this paper we have considered superintegrable systems in two-dimensional real Euclidean space, and shown that they possess hidden symmetries leading to linearization.

Other superintegrable systems have been found in two-dimensional non-Euclidean spaces, i.e. in two-dimensional space with non-constant curvature. Examples of such systems are the Perlick system [25], the Taub-NUT system [26], superintegrable systems for the Darboux space of Type I [11], and others [27], [28].

In a forthcoming paper [10] we will show that also superintegrable systems in non-Euclidean space can be reduced to linear equations by means of their hidden symmetries.

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